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On approximation numbers of Sobolev embeddings of weighted function spaces

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Abstract

We investigate asymptotic behaviour of approximation numbers of Sobolev embeddings between weighted function spaces of Sobolev–Hardy–Besov type with polynomials weights. The exact estimates are proved in almost all cases.

Keywords: Sobolev embeddings; Approximation numbers; Weighted function spaces

Approximation numbers of Sobolev embeddings between functions spaces of Sobolev–Besov–Hardy type have been studied in recent years by several authors. The approximation numbers of Sobolev embeddings of function spaces defined on bounded domains were studied by Edmunds and Triebel; cf. [6–8]. Later some of their estimates were improved by Caetano; cf. [3]. For weighted function spaces the counterpart of the theory was studied by Mynbaev and Otel'baev [20] in the case of fractional Sobolev spaces, and then more generally by Haroske, cf. [11], and by Caetano, cf. [3]. Some later results are described in [12]. In contrast to the function spaces on bounded domains, where the exact estimates were proved in almost all cases, the estimates for weighted spaces are less final. The importance of the asymptotic behavior of approximation and entropy numbers of Sobolev embeddings for the spectral theory of operators is discussed in [8]; cf. also [5,15].

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The aim of this paper is to prove the exact estimates of the asymptotic behavior of the approximation numbers in the so-called not-limiting case. We improve some earlier results obtained by Haroske and Caetano. However, the method we use is quite different from that one used in [11,3]; therefore we present further how it works in all cases. It is essentially the same method that was used for investigation of asymptotic behavior of entropy numbers of the embeddings; cf. [16].

Following Haroske [11] and Caetano [3], we consider the spaces with polynomial weights $w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha > 0$. The Sobolev embeddings of weighted Besov spaces,

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d), \tag{1}$$

$$-\infty < s_1 < s_0 < \infty, \qquad 1 \leqslant p_0 \leqslant p_1 \leqslant \infty, \qquad 1 \leqslant q_0, q_1 \leqslant \infty, \tag{2}$$

hold if $\delta = s_0 - s_1 - d(\frac{1}{p_0} - \frac{1}{p_1}) > 0$. The not-limiting case means that $\delta \neq \alpha$. We also consider the case $p_1 < p_0$.

We say that $a \sim b$ if there exists a constant $c > 0$ (independent of relevant parameters) such that

$$c^{-1}\, a \leqslant b \leqslant c\, a.$$

Let a_k be k-approximation numbers of the embeddings (1). We prove that

$$a_k \sim k^{-\varkappa}, \tag{3}$$

where $\varkappa > 0$ is a positive constant depending on p_0, p_1, s_0, s_1, and α; cf. Theorem 17. As a consequence we can prove the exact estimates of approximation numbers of the Sobolev embeddings on bounded C^∞ domain if $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$ and $\delta = \frac{d}{\min(p_0', p_1)}$. This seems to be the first proof of the estimates.

We get also some estimates in the limiting case $\alpha = \delta$, but in this case our estimates are not exact. Namely, we get

$$ck^{-\varkappa} \leqslant a_k \leqslant C_\varepsilon k^{-\varkappa}(1 + \log k)^{\varkappa+1+\varepsilon}, \qquad \varepsilon > 0. \tag{4}$$

The paper is organized as follows. In Section 1 we collect definitions and preliminary results needed later. In Section 2 we regard the approximation numbers of embeddings of related sequence spaces. The main result is formulated and proved in the last section.

1. Preliminaries

1.1. Approximation numbers

We recall the definitions of approximation numbers and corresponding operator ideal quasi-norms, which will be used widely in the paper. We refer to books by Carl and Stephani [4] and Pietsch [21] for details, proofs, and more information.

Let B_0 and B_1 be two complex Banach spaces and let $T : B_0 \to B_1$ be a bounded linear operator.

Definition 1. The *kth approximation number* $a_k(T)$ of the operator $T : B_0 \to B_1$ is the infimum of all numbers $\|T - A\|$; where A runs over the collection of all continuous linear operators $A : B_0 \to B_1$ of rank smaller than k.

Approximation numbers $a_k(T)$ form a decreasing sequence with $a_1(T) = \|T\|$. If the sequence converges to zero then the operator T is compact. The opposite implication is not true. It may happen that $\lim_{k\to\infty} a_k(T) > 0$ for some compact T if B_1 fails to have the approximation property. The approximation numbers have in particular the following properties:

— (additivity) $a_{n+k-1}(T_1 + T_2) \leqslant a_k(T_1) + a_n(T_2)$,
— (multiplicativity) $a_{n+k-1}(T_1 T_2) \leqslant a_k(T_1) a_n(T_2)$,
— (rank property) $a_k(T) = 0 \Leftrightarrow \text{rank}\,(T) < k$.

For a positive real number s we put

$$L^{(a)}_{s,\infty}(T) := \sup_{k\in\mathbb{N}} k^{1/s}\, a_k(T). \tag{5}$$

The expression $L^{(a)}_{s,\infty}(T)$ is an example of an *operator ideal quasi-norm*. This means in particular that there exists a number $0 < \varrho \leqslant 1$ such that

$$L^{(a)}_{s,\infty}\left(\sum_j T_j\right)^{\varrho} \leqslant \sum_j L^{(a)}_{s,\infty}(T_j)^{\varrho}, \tag{6}$$

for any sequence of operators $T_j : B_0 \to B_1$; cf. König [15, 1.c.5].

The following lemma concerning approximation numbers of embeddings of finite-dimensional complex ℓ_p-spaces can be found in Edmund's and Triebel's book [8, Corollary 3.2.3], but it is essentially due to Gluskin [9]; cf. also [20].

Lemma 2. *Let* $N \in \mathbb{N}$ *and* $k \leqslant \frac{N}{4}$.

(i) *If* $1 \leqslant p_0 \leqslant p_1 \leqslant 2$ *or* $2 \leqslant p_0 \leqslant p_1 \leqslant \infty$ *then*

$$a_k\Big(\text{id} : \ell^N_{p_0} \mapsto \ell^N_{p_1}\Big) \sim 1.$$

(ii) *If* $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$, $(p_0, p_1) \neq (1, \infty)$ *then*

$$a_k\Big(\text{id} : \ell^N_{p_0} \mapsto \ell^N_{p_1}\Big) \sim \min\big(1, N^{1/t} k^{-1/2}\big),$$

where $\frac{1}{t} = \frac{1}{\min(p_0', p_1)}$.

For $p_1 < p_0$ the corresponding approximation numbers were calculated by Pietsch [21, p. 109].

Lemma 3. *Let* $1 \leqslant p_1 < p_0 \leqslant \infty$. *Then*

$$a_k\Big(\text{id} : \ell^N_{p_0} \mapsto \ell^N_{p_1}\Big) = \big(N - k + 1\big)^{1/p_1 - 1/p_0}, \qquad k = 1, \ldots, N.$$

1.2. Weighted function spaces

We assume that the reader is acquainted with the definitions and basic properties of Besov and Triebel–Lizorkin spaces. Triebel's books [26,27] are the classical references here, but one can also consult [8] and many other books. In this section we recall a definition and a few of the properties of weighted function spaces.

In what follows we will be interested in the function spaces with polynomial weights

$$w_\alpha(x) = \left(1+|x|^2\right)^{\alpha/2}, \qquad \alpha > 0. \tag{7}$$

Definition 4. Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $s \in \mathbb{R}$. Let w_α be the above weight function. Then we put

$$B^s_{p,q}(\mathbb{R}^d, w) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f|B^s_{p,q}(\mathbb{R}^d, w)\| = \|fw|B^s_{p,q}(\mathbb{R}^d)\| < \infty \right\},$$

$$F^s_{p,q}(\mathbb{R}^d, w) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f|F^s_{p,q}(\mathbb{R}^d, w)\| = \|fw|F^s_{p,q}(\mathbb{R}^d)\| < \infty \right\},$$

for $p \neq \infty$ in the F-case.

Remark 5. It follows immediately from the definition that an operator $f \mapsto wf$ is an isomorphic mapping from $B^s_{p,q}(\mathbb{R}^d, w)$ onto $B^s_{p,q}(\mathbb{R}^d)$ and from $F^s_{p,q}(\mathbb{R}^d, w)$ onto $F^s_{p,q}(\mathbb{R}^d)$.

Remark 6. There are different ways to introduce weighted function spaces; cf., e.g., Triebel [26], Löfström [18], Schmeisser and Triebel [24], Buy et al. [2], Edmunds and Triebel [8], or Rychkov [23]. In the cited works the authors start with a Fourier-analytic definition. Under certain extra conditions on the weights these different approaches coincide; cf. [8,18,23, 24].

It follows from the definition of the spaces that in studying embeddings between weighted spaces it is sufficient to consider embeddings between weighted and unweighted spaces; cf. Remark 5. The following characterization of the compactness of embeddings was proved in [16]; cf. also [13].

Proposition 7. *Let* $1 \leqslant p_0, p_1 \leqslant \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, $-\infty < s_1 < s_0 < \infty$, *and* $\alpha > 0$. Let $\delta = s_0 - s_1 - \frac{d}{p_0} + \frac{d}{p_1}$. *The embedding* $B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ *is compact if and only if* $\min(\alpha, \delta) > d \max\left(\frac{1}{p_1} - \frac{1}{p_0}, 0\right)$.

The similar theorem holds also for $F^s_{p,q}$-spaces. The symbol $\hookrightarrow$ will be used for continuous embedding.

1.3. Wavelet characterizations of weighted Besov spaces

Wavelet bases in Besov spaces are a well-developed concept. In the case of unweighted spaces we refer to the monographs by Meyer [19] and Wojtaszczyk [29] and the article by Bourdaud [1] Here we are interested in the wavelet bases in the weighted cases. We quote the wavelet characterization of weighted Besov spaces proved in [16], but cf. also [25] and [22], where more general weights are considered.

First of all we need to fix some notations. By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, and by $\mathbb{Z}^d$ the set of all lattice points in $\mathbb{R}^d$ having integer components.

Let $\widetilde{\phi}$ be an orthogonal scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let $\widetilde{\psi}$ be an associated wavelet. Then the tensor product ansatz yields a scaling function ϕ and associated wavelets $\psi_1, \ldots, \psi_{2^d-1}$, all defined now on $\mathbb{R}^d$. We assume that

$$\widetilde{\phi} \in C^{N_1} \qquad \text{and} \qquad \operatorname{supp} \widetilde{\phi} \subset [-N_2,\ N_2]$$

for certain natural numbers N_1 and N_2. This implies that

$$\phi,\ \psi_i \in C^{N_1} \qquad \text{and} \qquad \operatorname{supp} \phi,\ \operatorname{supp} \psi_i \subset [-N_3,\ N_3]^d, \quad i=1, \ldots, 2^d - 1. \tag{8}$$

We shall use the standard abbreviations

$$\phi_{j,\ell}(x) = 2^{jd/2}\, \phi(2^j x - \ell) \qquad \text{and} \qquad \psi_{i,j,\ell}(x) = 2^{jd/2}\, \psi_i(2^j x - \ell). \tag{9}$$

Apart from function spaces with weights, we introduce sequence spaces with weights. If w is a given continuous weight function then $w(j, \ell) = w(2^{-j}\ell)$ for $j \in \mathbb{N}_0$ and $\ell \in \mathbb{Z}^d$. Let $1 \leqslant p, q \leqslant \infty$. We put

$$\ell_q^*\Big(2^{js}\ell_p(w)\Big) := \Bigg\{ \lambda = \{\lambda_{i,j,\ell}\}_{i,j,\ell} : \quad \lambda_{i,j,\ell} \in \mathbb{C},$$
$$= \|\, \lambda \,|\ell_q^*\Big(2^{js}\ell_p(w)\Big)\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_{i=1}^{2^d-1} \sum_{\ell \in \mathbb{Z}^d} |\lambda_{i,j,\ell}\, w(j,\ell)|^p \right)^{q/p} \right)^{1/q} < \infty \Bigg\} \tag{10}$$

and

$$\ell_q\Big(2^{js}\ell_p(w)\Big) := \Bigg\{ \lambda = \{\lambda_{j,\ell}\}_{j,\ell} : \quad \lambda_{j,\ell} \in \mathbb{C},$$
$$\|\, \lambda \,|\ell_q\Big(2^{js}\ell_p(w)\Big)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_{\ell \in \mathbb{Z}^d} |\lambda_{j,\ell}\, w_{j,\ell}|^p \right)^{q/p} \right)^{1/q} < \infty \Bigg\}. \tag{11}$$

For smooth weights and compactly supported wavelets it makes sense to consider the Fourier-wavelet coefficients of functions $f \in L_p(w)$ with respect to such an orthonormal basis. The following theorem was proved in [16], but one can also consult [25].

Theorem 8. *Let ϕ be a scaling function and let ψ_i, $i = 1, \ldots, 2^d - 1$; be the corresponding wavelets satisfying* (8). *Let $1 \leqslant p, q \leqslant \infty$ and let $0 < s < N_1$. Then a function*

$f \in L_p(\mathbb{R}^d, w_\alpha)$ belongs to $B^s_{p,q}(\mathbb{R}^d, w_\alpha)$, $\alpha > 0$, if and only if

$$\| f \,|B^s_{p,q}(\mathbb{R}^d, w_\alpha)\|^{\clubsuit} = \left(\sum_{\ell \in \mathbb{Z}^d} |\langle f, \phi_{0,\ell}\rangle \, w_\alpha(\ell)|^p \right)^{1/p}$$

$$+ \sum_{i=1}^{2^d-1} \left\{ \sum_{j=0}^{\infty} 2^{j\left(s+d(\frac{1}{2}-\frac{1}{p})\right)q} \left(\sum_{\ell \in \mathbb{Z}^d} |\langle f, \psi_{i,j,\ell}\rangle \, w_\alpha(2^{-j}\ell)|^p \right)^{q/p} \right\}^{1/q} < \infty. \quad (12)$$

Furthermore, $\| f \,|B^s_{p,q}(\mathbb{R}^d, w_\alpha)\|^{\clubsuit}$ may be used as an equivalent norm in

$B^s_{p,q}(\mathbb{R}^d, w_\alpha)$.

Remark 9. There is another way of discretizing of the function spaces useful for investigation of properties of entropy and approximation numbers of Sobolev embeddings. This is so-called quarkonial decomposition. The method was developed by H. Triebel in [28].

2. Approximation numbers of embeddings of some sequence spaces

We start with the following lemma which is simply a corollary of Lemma 2; cf. also [3].

Lemma 10. *Let $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$, $(p_0, p_1) \neq (1, \infty)$, and $N = 1, 2, 3, \ldots$. Then there is a positive constant C independent of N and k such that*

$$a_k\left(\mathrm{id} : \ell^N_{p_0} \mapsto \ell^N_{p_1}\right) \leqslant C \begin{cases} 1 & \text{if } k \leqslant N^{2/t}, \\ N^{1/t} k^{-1/2} & \text{if } N^{2/t} < k \leqslant N, \\ 0 & \text{if } k > N, \end{cases} \quad (13)$$

where $\frac{1}{t} = \max\left(\frac{1}{p_1}, \frac{1}{p_0'}\right)$.

Proof. The last estimate, for $k > N$, is obvious. To prove the others we regard the diagram

$$\begin{array}{ccc} \ell^N_{p_0} & \xrightarrow{\ S\ } & \ell^{4N}_{p_0} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ \ell^N_{p_1} & \xleftarrow{\ T\ } & \ell^{4N}_{p_1} \end{array}$$

where

$$S(\lambda_1, \ldots .\lambda_N) = (\lambda_1, \ldots, \lambda_N, 0, \ldots, 0) \quad \text{and}$$
$$T(\lambda_1, \ldots, \lambda_{4N}) = (\lambda_1, \ldots, \lambda_N).$$

Both the norms $\|S\|$ and $\|T\|$ are equal to 1; therefore $a_k(\mathrm{id}) \leqslant a_k(\mathrm{Id})$. Now the result follows from Lemma 2. $\square$

Proposition 11. *Suppose* $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$, $(p_0, p_1) \neq (1, \infty)$ *and* $\delta \neq \alpha$. *We put*

$$\varkappa = \begin{cases} \frac{\min(\alpha,\delta)}{d} + \frac{1}{2} - \frac{1}{\min(p_0', p_1)} & \text{if } \min(\alpha, \delta) > \frac{d}{\min(p_0', p_1)}, \\ \frac{\min(\alpha,\delta)}{d} \cdot \frac{\min(p_0', p_1)}{2} & \text{if } \min(\alpha, \delta) \leqslant \frac{d}{\min(p_0', p_1)}. \end{cases} \tag{14}$$

Then

$$a_k\Big(\mathrm{id} : \ell_{q_0}\big(2^{j\delta}\ell_{p_0}(w_\alpha)\big) \mapsto \ell_{q_1}\big(\ell_{p_1}\big)\Big) \quad \sim \quad k^{-\varkappa}. \tag{15}$$

Proof. *Step 1*. Preparations. We put

$$B_0 = \ell_{q_0}\big(2^{j\delta}\ell_{p_0}(w_\alpha)\big) \qquad \text{and} \qquad B_1 = \ell_{q_1}\big(\ell_{p_1}\big).$$

Let

$$\Lambda := \{\lambda = (\lambda_{j,\ell}) : \quad \lambda_{j,\ell} \in \mathbb{C}, \quad j \in \mathbb{N}_0,\ \ell \in \mathbb{Z}^d\}.$$

Let $I_{j,i} \subset \mathbb{N}_0 \times \mathbb{Z}^d$ be such that

$$I_{j,0} := \{(j, \ell) : \quad |\ell| \leqslant 2^j\}, \quad j \in \mathbb{N}_0, \tag{16}$$

$$I_{j,i} := \{(j, \ell) : \quad 2^{j+i-1} < |\ell| \leqslant 2^{j+i}\}, \quad i \in \mathbb{N}, \quad j \in \mathbb{N}_0. \tag{17}$$

Further, let $P_{j,i} : \Lambda \mapsto \Lambda$ be the canonical projection with respect to $I_{j,i}$; i.e., for $\lambda \in \Lambda$ we put

$$(P_{j,i}\lambda)_{u,v} := \begin{cases} \lambda_{u,v} & (u, v) \in I_{j,i}, \\ 0 & \text{otherwise}, \end{cases} \qquad u \in \mathbb{N}_0, \quad v \in \mathbb{Z}^d.$$

Observe that

$$M_{j,i} := |I_{j,i}| \sim 2^{(j+i)d}, \tag{18}$$

$$w_\alpha(2^{-j}\ell) \sim 2^{\alpha i} \qquad \text{if} \quad (j, \ell) \in I_{j,i}, \tag{19}$$

$$\mathrm{id}_\Lambda = \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} P_{j,i}. \tag{20}$$

Monotonicity arguments and elementary properties of the approximation numbers yield

$$\begin{aligned} a_k\Big(P_{j,i} : B_0 \mapsto B_1\Big) &\leqslant \frac{1}{\inf_{\ell \in I_{j,i}} w(2^{-j}\ell)} 2^{-j\delta} a_k\Big(\mathrm{id} : \ell_{p_0}^{M_{j,i}} \mapsto \ell_{p_1}^{M_{j,i}}\Big) \\ &\leqslant c\, 2^{-j\delta - i\alpha} a_k\Big(\mathrm{id} : \ell_{p_0}^{M_{j,i}} \mapsto \ell_{p_1}^{M_{j,i}}\Big). \end{aligned} \tag{21}$$

Step 2. To shorten the notation we put $\frac{1}{s} = \frac{1}{r} + \frac{1}{2}$ for any $s > 0$. Using (5) and (21) we find

$$L_{s,\infty}^{(a)}(P_{j,i}) \leqslant c\, 2^{-j\delta - i\alpha} L_{s,\infty}^{(a)}(\mathrm{id} : \ell_{p_0}^{M_{j,i}} \mapsto \ell_{p_1}^{M_{j,i}}). \tag{22}$$

The characterization of the asymptotic behavior of the approximation numbers of id : $\ell_{p_0}^N \mapsto \ell_{p_1}^N$, cf. (13), and (18) imply that

$$L_{2,\infty}^{(a)}(\mathrm{id} : \ell_{p_0}^{M_{j,i}} \mapsto \ell_{p_1}^{M_{j,i}}) \leqslant C\, 2^{d(j+i)/t}, \tag{23}$$

$$L_{s,\infty}^{(a)}(\mathrm{id} : \ell_{p_0}^{M_{j,i}} \mapsto \ell_{p_1}^{M_{j,i}}) \leqslant C\, 2^{d(j+i)(\frac{1}{t}+\frac{1}{r})}, \qquad \text{if} \quad \frac{1}{s} > \frac{1}{2}, \tag{24}$$

$$L_{s,\infty}^{(a)}(\mathrm{id} : \ell_{p_0}^{M_{j,i}} \mapsto \ell_{p_1}^{M_{j,i}}) \leqslant C\, 2^{d(j+i)(\frac{1}{t}+\frac{2}{rt})}, \qquad \text{if} \quad \frac{1}{2} > \frac{1}{s} > 0. \tag{25}$$

For a given $M \in \mathbb{N}_0$ let

$$P := \sum_{m=0}^{M} \sum_{j+i=m} P_{j,i} \qquad \text{and} \qquad Q := \sum_{m=M+1}^{\infty} \sum_{j+i=m} P_{j,i}. \tag{26}$$

Substep 2.1. The estimation of $a_k(P : B_0 \mapsto B_1)$. Let $\frac{1}{s} > \frac{1}{2}$. Then the formulae (22), (24) and (6) yield

$$\begin{aligned} L_{s,\infty}^{(a)}(P)^{\varrho} &\leqslant \sum_{m=0}^{M} \sum_{j+i=m} L_{s,\infty}^{(a)}(P_{j,i})^{\varrho} \\ &\leqslant c_1 \sum_{m=0}^{M} \sum_{j+i=m} 2^{-\varrho(j\delta+i\alpha)}\, 2^{\varrho m d(\frac{1}{t}+\frac{1}{r})} \\ &\leqslant c_2 \begin{cases} \sum_{m=0}^{M} 2^{\varrho m d(\frac{1}{r}+\frac{1}{t}-\frac{\alpha}{d})} & \text{if } \alpha < \delta \\ \sum_{m=0}^{M} 2^{\varrho m d(\frac{1}{r}+\frac{1}{t}-\frac{\delta}{d})} & \text{if } \alpha > \delta \\ \sum_{m=0}^{M} (m+1) 2^{\varrho m d(\frac{1}{r}+\frac{1}{t}-\frac{\alpha}{d})} & \text{if } \alpha = \delta. \end{cases} \end{aligned} \tag{27}$$

For $\delta > \alpha$ we choose r such that $d\left(\frac{1}{r}+\frac{1}{t}\right) - \alpha > 0$. Then (27) implies that

$$L_{s,\infty}^{(a)}(P) \leqslant c\, 2^{dM(\frac{1}{r}+\frac{1}{t}-\frac{\alpha}{d})}. \tag{28}$$

For $\delta < \alpha$ we choose r such that $d\left(\frac{1}{r}+\frac{1}{t}\right) - \delta > 0$. Then (27) implies that

$$L_{s,\infty}^{(a)}(P) \leqslant c\, 2^{dM(\frac{1}{r}+\frac{1}{t}-\frac{\delta}{d})}. \tag{29}$$

In view of (5), the inequalities (28) and (29) imply that

$$a_{2^{dM}}(P : B_0 \mapsto B_1) \leqslant c_3\, 2^{dM\left(\frac{1}{t}-\frac{1}{2}-\frac{\min(\alpha,\delta)}{d}\right)}. \tag{30}$$

So by the standard argument

$$a_k(P : B_0 \mapsto B_1) \leqslant c_3\, k^{-(\frac{\min(\alpha,\delta)}{d}+\frac{1}{2}-\frac{1}{t})} \tag{31}$$

for any $k \in \mathbb{N}$, if $\alpha \neq \delta$.

Substep 2.2. The estimate of $a_k(Q)$ from above. First we assume that $\mu = \min(\alpha, \delta) > \frac{d}{t}$. We proceed as in (27) and obtain by (23) that

$$L_{2,\infty}^{(a)}(Q)^{\varrho} \leqslant c_1 \sum_{m=M+1}^{\infty} 2^{\varrho m d(\frac{1}{t}-\frac{\mu}{d})} \times \begin{cases} 1 & \text{if } \alpha \neq \delta, \\ m+1 & \text{if } \alpha = \delta. \end{cases} \tag{32}$$

Since $\mu > \frac{d}{t}$ the formula (32) implies that

$$L^{(a)}_{2,\infty}(Q) \leqslant c\, 2^{dM(\frac{1}{t}-\frac{\mu}{d})}. \tag{33}$$

In view of (5), the inequality (33) implies that

$$a_{2^{Md}}(Q : B_0 \mapsto B_1) \leqslant c_3\, 2^{Md(\frac{1}{t}-\frac{1}{2}-\frac{\min(\alpha,\delta)}{d})} \tag{34}$$

and in consequence

$$a_k(Q : B_0 \mapsto B_1) \leqslant c_3\, k^{-(\frac{\min(\alpha,\delta)}{d}+\frac{1}{2}-\frac{1}{t})} \tag{35}$$

for any $k \in \mathbb{N}$.

Substep 2.3. Now let $\mu = \min(\alpha, \delta) \leqslant \frac{d}{t}$. We choose $-\frac{1}{2} < \frac{1}{r} < 0$, such that $0 < \frac{1}{t} + \frac{2}{tr} < \frac{\mu}{d} \leqslant \frac{1}{t}$. Then (25) gives

$$\begin{aligned} L^{(a)}_{s,\infty}(Q)^{\varrho} &\leqslant \sum_{m=M}^{\infty} \sum_{j+i=m} L^{(a)}_{s,\infty}(P_{j,i})^{\varrho} \\ &\leqslant c_1 \sum_{m=M}^{\infty} \sum_{j+i=m} 2^{-\varrho(j\delta+i\alpha)}\, 2^{\varrho m d(\frac{1}{t}+\frac{2}{tr})} \\ &\leqslant c_2 \sum_{m=M}^{\infty} 2^{\varrho m d(\frac{1}{t}+\frac{2}{tr}-\frac{\mu}{d})} \times \begin{cases} 1 & \text{if } \alpha \neq \delta \\ m+1 & \text{if } \alpha = \delta. \end{cases} \end{aligned} \tag{36}$$

Thus

$$L^{(a)}_{s,\infty}(Q) \leqslant c\, 2^{dM(\frac{1}{t}+\frac{2}{tr}-\frac{\mu}{d})} \tag{37}$$

if $\delta \neq \alpha$.

We put $k = [2^{Md\frac{2}{t}}]$. Then in view of (5), the inequality (37) implies

$$a_k(Q : B_0 \mapsto B_1) \leqslant c\, 2^{dM\frac{2}{t}(\frac{1}{2}+\frac{1}{r}-\frac{t\mu}{2d})} k^{-\frac{1}{r}-\frac{1}{2}} \leqslant c k^{-\frac{\mu}{d}\frac{t}{2}}. \tag{38}$$

By a standard argument (38) can be extended to any positive integer k.

Step 3. If $\mu > \frac{d}{t}$ then the proposition follows immediately from (31) and (35). If $\mu \leqslant \frac{d}{t}$ then $\frac{\mu}{d} + \frac{1}{2} - \frac{1}{t} \leqslant \frac{\mu}{d} \cdot \frac{t}{2}$. So the proposition follows from (31) and (38). This proves the estimates from above.

Step 4. Now we prove the estimate from below. We regard the following commutative diagram

$$\begin{array}{ccc} \ell^N_{p_0} & \xrightarrow{\;S\;} & \ell_{q_0}\left(2^{j\delta}\ell_{p_0}(w_\alpha)\right) \\ \text{Id}\downarrow & & \downarrow\text{id} \\ \ell^N_{p_1} & \xleftarrow{\;T\;} & \ell_{q_1}\left(\ell_{p_1}\right) \end{array} \tag{39}$$

Thus

$$a_k(\text{Id}) \leqslant \|S\|\, \|T\|\, a_k(\text{id}). \tag{40}$$

The definition of S and T as well as the value of N will depend on the given values of p_0, p_1, δ, and α. Let $v = (v_1, \dots, v_N)$ and $\lambda = (\lambda_{j,i})$.

(i) Let $\frac{d}{t} < \delta \leqslant \alpha$. We take $N = M_{k,0} = |I_{k,0}| \sim 2^{kd}$, $k \in \mathbb{N}$, $k \leqslant \frac{2}{d}$; cf. (16) and (18). Let $\varphi : \{1, \dots, N\} \to I_{k,0}$ be a bijection. We define

$$\big(S(v)\big)_{j,i} = \begin{cases} v_{\varphi^{-1}(i)} & \text{if} \quad (j,i) \in I_{k,0}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\big(T(\lambda)\big)_i = \lambda_{k,\varphi(i)}, \quad 1 \leqslant i \leqslant N.$$

Then $\|T\| = 1$ and $\|S\| \leqslant C 2^{k\delta}$, cf. (18) and (19). In consequence by Lemma 2 and (40) we get

$$C\, 2^{(dk-2)(\frac{1}{t}-\frac{1}{2})} \leqslant C_1 N^{\frac{1}{t}} 2^{-(dk-2)\frac{1}{2}} \leqslant 2^{k\delta} a_{2^{dk-2}}(\mathrm{id}). \tag{41}$$

This gives the estimate we are looking for.

(ii) Let $\frac{d}{t} < \alpha < \delta$. We take $N = M_{0,k} = |I_{0,k}| \sim 2^{kd}$, $k \in \mathbb{N}$, $k \leqslant \frac{2}{d}$, cf. (17) and (18). Let $\varphi : \{1, \dots N\} \to I_{0,k}$ be a bijection. We define

$$\big(S(v)\big)_{j,i} = \begin{cases} v_{\varphi^{-1}(i)} & \text{if} \quad (j,i) \in I_{0,k}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\big(T(\lambda)\big)_i = \lambda_{0,\varphi(i)}, \quad 1 \leqslant i \leqslant N.$$

Then $\|T\| = 1$ and $\|S\| \leqslant C 2^{k\alpha}$; cf. (18) and (19). In consequence by Lemma 2 and (40) we get

$$C\, 2^{(dk-2)(\frac{1}{t}-\frac{1}{2})} \leqslant 2^{k\alpha} a_{2^{dk-2}}(\mathrm{id}). \tag{42}$$

(iii) Let $\delta \leqslant \frac{d}{t}$ and $\delta < \alpha$. We choose the same N, S, and T as in the point (i) and put $\ell = [N^{2/t}]$. We have $\ell \leqslant N/4$ for sufficiently large N since $2 \leqslant t$. Moreover $N^{1/t} \ell^{-1/2} \sim 1$. So by Lemma 2 and (40) we get

$$C\, \ell^{-\frac{\delta t}{d2}} \leqslant C\, 2^{-k\delta} \leqslant a_\ell(\mathrm{id}). \tag{43}$$

(iv) Let $\alpha \leqslant \frac{d}{t}$ and $\alpha \leqslant \delta$. We choose the same N, S, and T as in point (ii). Arguments similar to the last one with $\ell = [N^{2/t}]$ give as

$$C\, \ell^{-\frac{\alpha t}{d2}} \leqslant C\, 2^{-k\alpha} \leqslant a_\ell(\mathrm{id}). \tag{44}$$

This finishes the proof. □

Remark 12. The method of the proof is not exact for the limiting case $\delta = \alpha$ but it gives some hints in this case also. Now we need additional information, about the relation between s and ϱ. Here one can always use $1/\varrho = 1/s + 1$; cf. [15, 1.c.5]. It turns out that with $\frac{1}{s} = \frac{\mu}{d} + \frac{1}{2} - \frac{1}{t} + \varepsilon$, $\varepsilon > 0$, the estimate

$$\begin{aligned} a_{2^{Md}}\Big(P : B_0 \mapsto B_1\Big) &\leqslant c\, 2^{-Md(\frac{\mu}{d}+\frac{1}{2}-\frac{1}{t})} M^{1/\varrho} \\ &= c\, 2^{-Md(\frac{\mu}{d}+\frac{1}{2}-\frac{1}{t})} M^{\frac{\mu}{d}+\frac{1}{2}-\frac{1}{t}+1+\varepsilon} \end{aligned} \tag{45}$$

holds.

Similarly, if $\frac{\mu}{d} > \frac{1}{t}$, then we get the estimate

$$a_{2^{Md}}\Big(Q : B_0 \mapsto B_1\Big) \leqslant c\, 2^{Md(-\frac{\mu}{d}-\frac{1}{2}+\frac{1}{t})}\, M^{1/\varrho} = c\, 2^{Md(-\frac{\mu}{d}-\frac{1}{2}+\frac{1}{t})}\, M^{3/2}.$$

Both together yield

$$a_k\Big(\mathrm{id} : B_0 \mapsto B_1\Big) \leqslant c_\varepsilon\, k^{-(\frac{\mu}{d}+\frac{1}{2}-\frac{1}{t})}\, (1+\log k)^{\frac{\mu}{d}+\frac{1}{2}-\frac{1}{t}+1+\varepsilon}, \tag{46}$$

where $\frac{\mu}{d} > \frac{1}{t}$, and $\varepsilon > 0$ may be chosen arbitrarily small.

In the similar way, if $\frac{\mu}{d} \leqslant \frac{1}{t}$ then taking $\frac{1}{s} = \frac{\mu}{d} \cdot \frac{t}{2} - \varepsilon$, $\varepsilon > 0$, we get the estimates

$$a_{2^{Md}}\Big(Q : B_0 \mapsto B_1\Big) \leqslant c\, 2^{-Md\frac{\mu}{d}\cdot\frac{t}{2}}\, M^{1/\varrho} = c\, 2^{-Md\frac{\mu}{d}\cdot\frac{t}{2}}\, M^{\frac{\mu}{d}\cdot\frac{t}{2}+1-\varepsilon}.$$

The last estimates and (45) yield

$$a_k\Big(\mathrm{id} : B_0 \mapsto B_1\Big) \leqslant c_\varepsilon\, k^{-(\frac{\mu}{d}\cdot\frac{t}{2})}\, (1+\log k)^{\frac{\mu}{d}\cdot\frac{t}{2}+1+\varepsilon}, \tag{47}$$

where $\frac{\mu}{d} \leqslant \frac{1}{t}$, and $\varepsilon > 0$ may be chosen to be arbitrarily small.

The exponent of the logarithmic term is not optimal, but the estimate shows that a logarithmic term is the only additional term we can expect in the limiting case.

Proposition 13. *Suppose* $1 \leqslant p_0 \leqslant p_1 \leqslant 2$ *or* $2 \leqslant p_0 \leqslant p_1 \leqslant \infty$ *and* $\delta \neq \alpha$. *Then*

$$a_k\Big(\mathrm{id} : \ell_{q_0}\big(2^{j\delta}\ell_{p_0}(w_\alpha)\big) \mapsto \ell_{q_1}\big(\ell_{p_1}\big)\Big) \sim k^{-\varkappa}, \tag{48}$$

where

$$\varkappa = \frac{\min(\alpha, \delta)}{d}. \tag{49}$$

Proof. *Step 1* (Estimates from Above). We can deal with the proof in a way similar to that for the previous proposition, so we use the same notation. Now Lemma 2 implies that

$$L^{(a)}_{s,\infty}(\mathrm{id} : \ell^{M_{j,i}}_{p_0} \mapsto \ell^{M_{j,i}}_{p_1}) \leqslant M^{1/s}_{j,i} \leqslant C 2^{(j+i)\frac{d}{s}}. \tag{50}$$

So, using (22), (26), (6), and (50) we find that

$$\begin{aligned} L^{(a)}_{s,\infty}(P)^\varrho &\leqslant \sum_{m=0}^{M} \sum_{j+i=m} 2^{-\varrho(j\delta+i\alpha)}\, 2^{\varrho m\frac{d}{s}} \\ &\leqslant c_2 \sum_{m=0}^{M} 2^{\varrho md(\frac{d}{s}-\alpha)} \leqslant c_3\, 2^{\varrho dM(\frac{1}{s}-\alpha)} \end{aligned} \tag{51}$$

for $\delta > \alpha$ and $\frac{1}{s} > \frac{\alpha}{d}$. Thus

$$a_{2^{dM}}(P : B_0 \mapsto B_1) \leqslant c_3\, 2^{-dM\frac{\alpha}{d}}. \tag{52}$$

In the same way we prove that

$$a_{2^{dM}}(P : B_0 \mapsto B_1) \leqslant c_3\, 2^{-dM\frac{\delta}{d}} \tag{53}$$

if $\delta < \alpha$.

For the operator Q we have

$$L^{(a)}_{s,\infty}(Q)^{\varrho} \leqslant \sum_{m=M+1}^{\infty} \sum_{j+i=m} 2^{-\varrho(j\delta+i\alpha)}\, 2^{\varrho m \frac{d}{s}}$$

$$\leqslant c_2 \sum_{m=M+1}^{\infty} 2^{\varrho m d(\frac{d}{s}-\mu)} \leqslant c_3\, 2^{\varrho d M(\frac{1}{s}-\mu)}, \tag{54}$$

where $\mu = \min(\delta, \alpha)$ and $\frac{1}{s} < \frac{\mu}{d}$. Thus

$$a_{2^{dM}}(Q : B_0 \mapsto B_1) \leqslant c_3\, 2^{-dM\frac{\mu}{d}}. \tag{55}$$

Now (52), (53), and (55) give us the estimate from above.

Step 2 (Estimates from Below). We can deal with the estimates in a way similar to that in *Step 4* of the last proof. If $\delta < \alpha$ we take N, S, and T the same as in point (i). Using Lemma 2 we get

$$C \leqslant 2^{k\delta} a_{2^{dk-2}}(\mathrm{id}).$$

If $\alpha < \delta$ we should follow the point (ii). □

Remark 14. Once more the proof gives us an estimate for the limiting case $\delta = \alpha$. Now reasoning similar to that in Remark 12 leads to the estimate

$$a_k(\mathrm{id}) \leqslant C_\varepsilon\, k^{-\frac{\alpha}{d}} (1 + \log k)^{\frac{\alpha}{d}+1+\varepsilon},$$

where $\varepsilon > 0$.

Proposition 15. *Let* $1 \leqslant p_0, p_1 \leqslant \infty$, $\frac{1}{\tilde{p}} = \frac{\min(\alpha,\delta)}{d} + \frac{1}{p_0}$, *and* $\delta \neq \alpha$. *Suppose* $\tilde{p} < p_1 < p_0 \leqslant \infty$. *Then*

$$a_k\Big(\mathrm{id} : \ell_{q_0}\big(2^{j\delta}\ell_{p_0}(w_\alpha)\big) \mapsto \ell_{q_1}\big(\ell_{p_1}\big)\Big) \sim k^{-\varkappa}, \tag{56}$$

with

$$\varkappa = \frac{\min(\alpha, \delta)}{d} + \frac{1}{p_0} - \frac{1}{p_1}. \tag{57}$$

Proof. *Step 1* (Estimates from Above). We only sketch the proof since, once more, we can use the same reasoning. We put $\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_0}$. Now by Lemma 3

$$a_k\Big(\mathrm{id} : \ell_{p_0}^{M_{j,i}} \mapsto \ell_{p_1}^{M_{j,i}}\Big) \sim \big(M_{j,i} - k + 1\big)^{1/p}, \qquad k = 1, \ldots, M_{j,i}. \tag{58}$$

In consequence

$$L^{(a)}_{s,\infty}(\mathrm{id} : \ell^{M_{j,i}}_{p_0} \mapsto \ell^{M_{j,i}}_{p_1}) \leqslant M_{j,i}^{\frac{1}{s}+\frac{1}{p}}, \qquad \frac{1}{s} > 0. \tag{59}$$

So, using (22), (26), (6), and (59), we find that

$$\begin{aligned} L^{(a)}_{s,\infty}(P)^{\varrho} &\leqslant \sum_{m=0}^{M} \sum_{j+i=m} 2^{-\varrho(j\delta+i\alpha)}\, 2^{\varrho md(\frac{1}{s}+\frac{1}{p})} \\ &\leqslant c_2 \sum_{m=0}^{M} 2^{\varrho md(\frac{1}{s}+\frac{1}{p}-\frac{\mu}{d})} \leqslant c_3\, 2^{\varrho dM(\frac{1}{s}+\frac{1}{p}-\frac{\mu}{d})}, \end{aligned} \tag{60}$$

with $\mu = \min(\delta, \alpha)$ and $\frac{1}{s} > \frac{\mu}{d} - \frac{1}{p}$. Thus,

$$a_{2^{dM}}(P : B_0 \mapsto B_1) \leqslant c_3\, 2^{-dM(\frac{\mu}{d}-\frac{1}{p})}. \tag{61}$$

For the operator Q we have

$$\begin{aligned} L^{(a)}_{s,\infty}(Q)^{\varrho} &\leqslant \sum_{m=M+1}^{\infty} \sum_{j+i=m} 2^{-\varrho(j\delta+i\alpha)}\, 2^{\varrho md(\frac{1}{s}+\frac{1}{p})} \\ &\leqslant c_2 \sum_{m=M+1}^{\infty} 2^{\varrho md(\frac{1}{s}+\frac{1}{p}-\frac{\mu}{d})} \leqslant c_3\, 2^{\varrho dM(\frac{1}{s}+\frac{1}{p}-\frac{\mu}{d})} \end{aligned} \tag{62}$$

if $0 < \frac{1}{s} < \frac{\mu}{d} - \frac{1}{p}$. Such s can always be taken since by the assumptions $\frac{1}{p} < \frac{\mu}{d}$. Thus

$$a_{2^{dM}}(Q : B_0 \mapsto B_1) \leqslant c_3\, 2^{-dM(\frac{\mu}{d}-\frac{1}{p})}. \tag{63}$$

Now (61) and (63) give us the estimate from above.

Step 2 (Estimates from Below). Once more we follow Step 4 of the proof of Proposition 11, now using (58) instead of Lemma 2. □

Remark 16. For the limiting case $\delta = \alpha$ we now get the estimate

$$a_k(\mathrm{id}) \leqslant C_\varepsilon\, k^{-\frac{\alpha}{d}+\frac{1}{p_0}-\frac{1}{p_1}} (1+\log k)^{\frac{\alpha}{d}-\frac{1}{p_0}+\frac{1}{p_1}+1+\varepsilon},$$

where $\varepsilon > 0$.

3. Approximation numbers of Sobolev embeddings

We shall let $A^s_{p,q}(\mathbb{R}^d, w_\alpha)$ ($A^s_{p,q}(\mathbb{R}^d)$) stand for either $B^s_{p,q}(\mathbb{R}^d, w_\alpha)$ ($B^s_{p,q}(\mathbb{R}^d)$) or $F^s_{p,q}(\mathbb{R}^d, w_\alpha)$ ($F^s_{p,q}(\mathbb{R}^d)$), with the understanding that for the F-spaces we must have $p < \infty$. The main result of the paper reads as follows.

Theorem 17. *Let* $1 \leqslant p_0, p_1 \leqslant \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, *and* $-\infty < s_1 < s_0 < \infty$. *Let* $\alpha > 0$, $\delta = s_0 - s_1 - d(\frac{1}{p_0} - \frac{1}{p_1}) > 0$, *and* $\frac{1}{\tilde{p}} = \frac{\min(\alpha,\delta)}{d} + \frac{1}{p_0}$. *We assume that*

(a) $1 \leqslant p_0 \leqslant p_1 \leqslant \infty$ *or* $\tilde{p} < p_1 < p_0 \leqslant \infty$,
(b) $\alpha \neq \delta$.

Let a_k *denote the kth approximation number of the Sobolev embedding*

$$A^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_1}_{p_1,q_1}(\mathbb{R}^d).$$

Then

$$a_k \sim k^{-\varkappa},$$

where

(i) $\varkappa = \frac{\min(\alpha,\delta)}{d}$ *if* $1 \leqslant p_0 \leqslant p_1 \leqslant 2$ *or* $2 \leqslant p_0 \leqslant p_1 \leqslant \infty$,
(ii) $\varkappa = \frac{\min(\alpha,\delta)}{d} + \frac{1}{p_0} - \frac{1}{p_1}$ *if* $\tilde{p} < p_1 < p_0 \leqslant \infty$,
(iii) $\varkappa = \frac{\min(\alpha,\delta)}{d} + \frac{1}{2} - \frac{1}{\min(p_0', p_1)}$ if $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$, $(p_0, p_1) \neq (1, \infty)$ *and* $\min(\alpha, \delta) > \frac{d}{\min(p_0', p_1)}$,
(iv) $\varkappa = \frac{\min(\alpha,\delta)}{d} \cdot \frac{\min(p_0', p_1)}{2}$ *if* $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$, $(p_0, p_1) \neq (1, \infty)$ *and* $\min(\alpha, \delta) \leqslant \frac{d}{\min(p_0', p_1)}$.

Proof. If both $A^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_\alpha)$ and $A^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ are Besov spaces then the assertions follow from Theorem 8 and Proposition 11, Proposition 13, or Proposition 15, respectively.

The general upper estimates follow from the Besov case, the multiplicativity property of approximation numbers, and elementary embeddings

$$A^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_0}_{p_0,\infty}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_1}_{p_1,1}(\mathbb{R}^d) \hookrightarrow A^{s_1}_{p_1,q_1}(\mathbb{R}^d).$$

To prove lower estimates one should consider the following embeddings:

$$B^{s_0}_{p_0,1}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_1}_{p_1,\infty}(\mathbb{R}^d). \qquad \square$$

Remark 18. Let all the assumption of the theorem hold except assumption (b). If $\alpha = \delta$ then for any $\varepsilon > 0$ there exist constants $c > 0$ and $C_\varepsilon > 0$ such that

$$c\,k^{-\varkappa} \leqslant a_k \leqslant C_\varepsilon\, k^{-\varkappa}\,(1 + \log k)^{\varkappa+1+\varepsilon}. \tag{64}$$

The only exact estimate, known for the limiting case, was proved by Mynbaev and Otel'baev for Bessel potential spaces. In our notation their result reads

$$a_k\Big(F^{s_0}_{p_0,2}(\mathbb{R}^d, w_\alpha) \hookrightarrow F^{0}_{p_1,2}(\mathbb{R}^d)\Big) \sim k^{-\frac{\alpha}{d}}(1 + \log k)^{\frac{\alpha}{d}},$$

where

$$s_0 > 0\,, \qquad 1 < p_0 \leqslant p_1 \leqslant 2\,, \qquad 2 \leqslant p_0 \leqslant p_1 < \infty;$$

cf. [20, V, Section 3, Theorem 9]. Some other estimates, not exact but better then the estimate (64), can be found in [12]. So the only advantage of (64) is that the estimates holds with no additional assumption.

At the end we use this theorem to fill the only gap in estimation of approximation numbers of Sobolev embeddings on bounded domains.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^∞ boundary. Let $A^s_{p,q}(\Omega)$, be the Besov space or the Triebel–Lizorkin space on Ω defined by restrictions; cf. [8, Chap. 2.5]; [26, Chap. 4]; [27, Chap. 3].

Corollary 19. *Let a_k denote the kth approximation number of the Sobolev embedding*

$$A^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow A^{s_1}_{p_1,q_1}(\Omega).$$

Then

$$a_k \leqslant C\, k^{-\varkappa},$$

where

(i) $\varkappa = \frac{\delta}{d}$ *if* $1 \leqslant p_0 \leqslant p_1 \leqslant 2$ or $2 \leqslant p_0 \leqslant p_1 \leqslant \infty$,

(ii) $\varkappa = \frac{\delta}{d} + \frac{1}{p_0} - \frac{1}{p_1}$ *if* $\tilde{p} \leqslant p_1 < p_0 \leqslant \infty$,

(iii) $\varkappa = \frac{\delta}{d} + \frac{1}{2} - \frac{1}{\min(p_0', p_1)}$ *if* $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$, $(p_0, p_1) \neq (1, \infty)$ *and* $\delta > \frac{d}{\min(p_0', p_1)}$,

(iv) $\varkappa = \frac{\delta}{d} \cdot \frac{\min(p_0', p_1)}{2}$ *if* $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$, $(p_0, p_1) \neq (1, \infty)$ *and* $\delta \leqslant \frac{d}{\min(p_0', p_1)}$.

Proof. We choose $\alpha > \delta$. We prove the corollary for Besov spaces; the rest is a consequence of elementary embeddings. Let $Rf = f|_\Omega$ be a restriction operator. It is a bounded linear operator from $B^s_{p,q}(\mathbb{R}^d)$ onto $B^s_{p,q}(\Omega)$. There exist an extension bounded linear operator $S : B^s_{p,q}(\Omega) \mapsto B^s_{p,q}(\mathbb{R}^d)$ such that $RS = \mathrm{id}$; cf. Theorem 3.3.4 in [27]. Let $S_\alpha f = w_\alpha^{-1} Sf$ and $R_\alpha f = w_\alpha f|_\Omega$. We have the following commutative diagram:

$$\begin{array}{ccc} B^{s_0}_{p_0,q_0}(\Omega) & \xrightarrow{S_\alpha} & B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_\alpha) \\ \mathrm{id}\downarrow & & \downarrow\mathrm{Id} \\ B^{s_1}_{p_1,q_1}(\Omega) & \xleftarrow{R_\alpha} & B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \end{array}$$

The operators S_α and R_α are bounded and $R_\alpha S_\alpha = \mathrm{id}$. In consequence $a_k(\mathrm{id}) \leqslant C\, a_k(\mathrm{Id})$. Now the corollary follows immediately from the previous theorem. □

Almost all assertions of the last corollary are known. Points (i)–(iii) were proved by Edmunds and Triebel; cf. [8, Theorem 3.3.4]. Point (iv) was proved by Caetano with additional assumption that $\delta < \frac{d}{\min(p_0', p_1)}$; cf. [3]. To the best of our knowledge our proof is the first that also covers the case $\delta = \frac{d}{\min(p_0', p_1)}$. Since the estimates from below are known, cf. [8, Theorem 3.3.4], we have the following corollary.

Corollary 20. *Let* $1 \leqslant p_0 < 2 < p_1 \leqslant \infty$, $(p_0, p_1) \neq (1, \infty)$, *and* $\delta = \frac{d}{\min(p_0', p_1)}$. *Then*

$$a_k \sim k^{-1/2}.$$

Remark added in proof. (1) The method presented in the paper can be used also in the quasi-Banach case, $0 < p < 1$ or $0 < q < 1$. The quasi-Banach version of Lemma 2 was proved in [8, Theorem 3.2.2], with the additional assumption $p_1 < \infty$ if $p_0 < 1$. Also Lemma 3 can be extended to the quasi-Banach spaces with $0 < p_0 < p_1 \leqslant \infty$. The proof of the quasi-Banach case is the same as that for Banach spaces; cf. [21, Proposition 2.9.8]. The wavelet characterization of weighted Besov spaces with $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$ was recently proved by Haroske and Triebel in [14]. In the end the technique of operator ideal quasi-norms also works for operators acting between quasi-normed spaces. This fact was used recently in [17]. So we can repeat the calculations from Section 2 with $p_0 < 1$, $p_1 < \infty$, and $0 < q_0, q_1 \leqslant \infty$. In consequence Theorem 17 also holds in the quasi-Banach case with $p_1 < \infty$ and $p_0' = \infty$ if $0 < p_0 < 1$.

(2) The technique also works for other type of weights. In particular, we have recently proved the following theorem.

Theorem 21. *Let* $0 < p_0, p_1 \leqslant \infty$, $0 < q_0, q_1 \leqslant \infty$, *and* $-\infty < s_1 < s_0 < \infty$. *Let* $v_\alpha(x) = \left(1 + \log(1 + |x|^2)\right)^\alpha$, $x \in \mathbb{R}^d$, *and* $\alpha > 0$. *If* $s_0 - s_1 - d(\frac{1}{p_0} - \frac{1}{p_1}) > 0$ *and* $p_0 \leqslant p_1$ *then the embedding*

$$A^{s_0}_{p_0,q_0}(\mathbb{R}^d, v_\alpha) \hookrightarrow A^{s_1}_{p_1,q_1}(\mathbb{R}^d)$$

is compact and

$$a_k\left(A^{s_0}_{p_0,q_0}(\mathbb{R}^d, v_\alpha) \hookrightarrow A^{s_1}_{p_1,q_1}(\mathbb{R}^d)\right) \sim v_\alpha(k)^{-1}.$$

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